

ON PRIMES OF ORDINARY AND HODGE-WITT REDUCTION

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ABSTRACT. Jean-Pierre Serre has conjectured, in the context of abelian varieties, that there are infinitely primes of good ordinary reduction for a smooth, projective variety over a number field. We consider this conjecture and its natural variants. In particular we conjecture the existence of infinitely many primes of Hodge-Witt reduction (any prime of ordinary reduction is also a prime of Hodge-Witt reduction). The two conjectures are not equivalent but are related. We prove a precise relationship between the two; we prove several results which provide some evidence for these conjectures; we show that primes of ordinary and Hodge-Witt reduction can have different densities. We prove our conjecture on Hodge-Witt and ordinary reduction for abelian varieties with complex multiplication. We include here an unpublished joint result with C. S. Rajan (also independently established by Fedor Bogomolov and Yuri Zarhin by a different method) on the existence of primes of ordinary reductions for K3 surfaces; our proof also shows that for an abelian threefold over a number field there is a set of primes of positive density at which it has Hodge-Witt reduction (this is also a joint result with C. S. Rajan). We give a number of examples including those of Fermat hypersurfaces for which all the conjectures we make hold.

आधीचा प्रसाद घेत, पुढची ऐकीत साद
नादातच शीळ वाजवीत चाललो,
वाटले जसा फुलांफुलांत चाललो.

आ. रा. देशपांडे “अनिल”

CONTENTS

1. Introduction	2
1.1. Motivating question	2
Acknowledgment	4
2. Preliminaries	4
2.1. Ordinary varieties	4
2.2. Cartier Operator	4
2.3. de Rham-Witt cohomology	5
3. Primes of ordinary reduction for $K3$ surfaces	6
3.1. Serre's conjecture	6
3.2. Trace of Frobenius	6
3.3. Deligne Weil estimate	6
3.4. Katz-Messing theorem	7
3.5. Semi-simplicity of the crystalline Frobenius	7
3.6. Mazur's theorem	7
3.7. Crystalline torsion	7

4. Primes of Hodge-Witt reduction	9
4.1. Hodge-Witt reduction	9
4.2. Hodge-Witt Abelian varieties	10
4.3. Primes of Hodge-Witt reduction for abelian threefolds	10
4.4. Hodge-Witt reduction of CM abelian varieties	12
5. Non Hodge-Witt reduction	15
5.1. Hodge-Witt torsion	16
6. Some explicit examples	18
6.1. Fermat varieties	18
References	19

1. INTRODUCTION

1.1. Motivating question. Let k be a perfect field of characteristic $p > 0$. An abelian variety A over k is said to be *ordinary* if the p -rank of A is the maximum possible, namely equal to the dimension of A . The notion of ordinarity was extended by Mazur [30], to a smooth projective variety X over k , using notions from crystalline cohomology. A more general definition was given by Bloch-Kato [5] and Illusie-Raynaud [20] using coherent cohomology. With this definition it is easier to see that ordinarity is an open condition. Ordinary varieties tend to have special properties, for example the existence of canonical Serre-Tate lifting for ordinary abelian varieties to characteristic 0, and the comparison theorems between crystalline cohomology and p -adic étale cohomology can be more easily established for such varieties. In brief, ordinary varieties play a key role in the study of varieties in characteristic $p > 0$.

The motivating question for this paper, is a conjecture of Jean-Pierre Serre (Conjecture 3.1.1) formulated originally in the context of abelian varieties. The conjecture of Serre is the following: Let K be a number field, and let X be a smooth, projective variety defined over K . Then there is a positive density of primes of K , at which X acquires ordinary reduction. Ordinary varieties are *Hodge-Witt*, in that the de Rham-Witt cohomology groups $H^i(X, W\Omega_X^j)$ are finitely generated over W . The class of Hodge-Witt varieties strictly contains the class of ordinary varieties. Varieties which are Hodge-Witt have distinct geometrical properties from varieties which are non-Hodge-Witt. However we do not know how to read all the subtle information encoded in the de Rham-Witt cohomology yet. But an example of this behavior is implicit in the recent work of Chai-Conrad-Oort (see [8]). We note that the word ‘Hodge-Witt’ is not mentioned in loc. cit. but some of the lifting/non-lifting phenomenon studied in [8] are dependent on whether the abelian variety in question is Hodge-Witt or non-Hodge-Witt. Hence for the reader’s convenience we provide a translation between the point of view of [8] and our point of view in Theorem 4.2.4. Following Serre, we ask more generally, if given a smooth, projective variety over a number field K , do there exist infinitely many primes of Hodge-Witt reduction? This is our Conjecture 4.1.1. This conjecture does not appear to have studied before and we investigate it in section 4. We give some evidence for this conjecture in this paper. We note that Conjecture 3.1.1 of Serre and our Conjecture 4.1.1 are not equivalent as there exists a smooth, projective variety over \mathbb{Q} such that the sets of primes provided by Conjecture 3.1.1 and Conjecture 4.1.1 are of different densities (see Example 6.1.1). But the two Conjectures are almost equivalent: in Theorem 4.1.3 we

prove the precise relationship between the two conjectures. One of the main results is that this is true for abelian threefolds over number fields (this is joint work with C. S. Rajan—even though it is not stated in its present form in [21]).

Some of the results of this note were included section 6,7 of my 2001 preprint [21] with C. S. Rajan. However when that preprint was accepted for publication in the International Math. Res. Notices (see [22]), the referee felt that the result for K3 surfaces might be well-known to experts and at any rate the contents of sections 6,7 of that preprint be published independently as the results contained in these two sections were independent of the main results of that paper (namely construction of examples of Frobenius split non-ordinary varieties). Subsequent to this the authors lost interest in separate publication of the contents of these two sections. One of the important results of the aforementioned sections of [21] is about ordinary reductions of K3 surfaces over number fields (Theorem 3.7.3(1)). This result has also been proved by Fedor Bogomolov and Yuri Zarhin (see [6]) by a different method. We also note here that a proof of the result for a class of K3 surfaces was also given by Tankeev (see [46]) under somewhat restrictive hypotheses. Both the approaches (ours and that of [6]) are important. For a non-trivial geometric application of Theorem 3.7.3(1) see [7, 29, 2]. Our proof of Theorem 3.7.3(1) mirrors closely the proof given by Ogus for abelian surfaces. Our methods are different from those of [6] and [46] and proves, in addition to establishing the existence of primes of ordinary reduction for K3 surfaces, a crucial step (see Theorem 3.7.3(2)) in the proof of existence of primes of Hodge-Witt reduction for abelian threefolds (Theorem 4.3.1). This is certainly a new result of independent interest. The proofs of these two results are closely linked and it would seem artificial to separate them. This is our rationale for publishing these results.

For purposes of citation, we provide joint attributions to results obtained with C. S. Rajan; all the unattributed or uncited results are ours.

We have also added to the results of the aforementioned sections of [21], Theorem 4.1.3 which clarifies the precise relationship between Conjecture 3.1.1 and Conjecture 4.1.1 and included a number of examples which were not contained in the sections mentioned above, including examples which show that the two conjectures while related are not equivalent.

Another important addition in present paper are Theorem 4.4.4 and Theorem 4.4.5 in which we prove Conjecture 3.1.1 and Conjecture 4.1.1 for any abelian variety with Complex Multiplication. We show that if the CM field is galois over \mathbb{Q} then any prime of Hodge-Witt reduction is a prime of ordinary reduction, but if the CM field is non-galois over \mathbb{Q} , then the set of primes of Hodge-Witt reduction and the set of primes of ordinary reduction may not coincide and we give an example of a CM field (and hence an abelian variety with CM by this field) where the density of primes of Hodge-Witt reduction is $\frac{3}{8}$ while the set of primes of ordinary reduction is of density $\frac{1}{8}$.

In a different direction, Mustata-Srinivas and Mustata (see [35, 36]) have related the existence of weak ordinary reductions to invariants of singularities (of subvarieties) and this work has also led to some interest in ordinary varieties. We note, as an aside, that Corollary 4.4.7 implies that the conjecture of Mustata-Srinivas (see [35, 36, Conjecture 1.1]) is true for abelian varieties over number fields with CM (see Corollary 4.4.9). Since any abelian variety with CM is definable over a number field, we see that the conjecture of Mustata-Srinivas is true for any abelian variety with CM over any field of characteristic zero.

In the final section we study the presence of infinite torsion in the de Rham-Witt cohomology of varieties and discuss some questions and conjectures. In particular in Conjecture 5.1.5 we conjecture that for a smooth, projective variety X over a number field K , with $H^n(X, \mathcal{O}_X) \neq 0$, there exist infinitely many primes of K at which X has good, non-Hodge-Witt reduction. This conjecture includes, as a special case, the conjecture that there exists infinitely many primes of good, supersingular reduction for any elliptic curve over K (for $K = \mathbb{Q}$, this is the well-known theorem of [13]). Our conjecture also implies a stronger assertion: given any two elliptic curves over a number field K , there exist infinitely many primes of K at which both the curves have good supersingular reduction. It also includes the assertion that given any g non-isogenous elliptic curves over a number field, then there are infinitely many primes p such that at most $g - 2$ of the curves have ordinary reduction at these primes.

Acknowledgment. Some parts of this paper were mostly written while the author was at, or visiting, the Tata Institute of Fundamental Research and support and hospitality of the institute is gratefully acknowledged. We also thank Vikram Mehta for conversations. We thank C. S. Rajan for many conversations and also for allowing us to include our previously unpublished joint results here. We would also like to thank Karl Schwede and Kevin Tucker, organizers of the 2011 AIMS workshop on “Relating test ideals and multiplier ideals” for giving me an opportunity to speak on results of [21]. We decided to revise the unpublished sections of [21] because of this workshop and the support and hospitality of AIMS is gratefully acknowledged. I thank the anonymous referee for suggesting a number of improvements and also for pointing out a mistake in the earlier version of Proposition 4.4.2 (I had claimed a stronger assertion than what is now proved—but this is quite adequate for what is needed here) and for a careful reading of the manuscript.

2. PRELIMINARIES

2.1. Ordinary varieties. Let X be a smooth projective variety over a perfect field k of positive characteristic. Following Bloch-Kato [5] and Illusie-Raynaud [20], we say that X is ordinary if $H^i(X, B_X^j) = 0$ for all $i \geq 0$, $j > 0$, where

$$B_X^j = \text{image} \left(d : \Omega_X^{j-1} \rightarrow \Omega_X^j \right).$$

If X is an abelian variety, then it is known that this definition coincides with the usual definition [5]. By [19, Proposition 1.2], ordinarity is an open condition in the following sense: if $X \rightarrow S$ is a smooth, proper family of varieties parameterized by S , then the set of points s in S , such that the fiber X_s is ordinary is a Zariski open subset of S . Although the following proposition is well known, we present it here as an illustration of the power of this fact.

2.2. Cartier Operator. Let X be a smooth proper variety over a perfect field of characteristic $p > 0$, and let F_X (or F) denote the absolute Frobenius of X . We recall a few basic facts about Cartier operators from [17]. The first fact we need is that we have a fundamental exact sequence of locally free sheaves

$$(2.2.1) \quad 0 \rightarrow B_X^i \rightarrow Z_X^i \xrightarrow{C} \Omega_X^i \rightarrow 0,$$

where Z_X^i is the sheaf of closed i -forms, where C is the Cartier operator. The existence of this sequence is the fundamental theorem of Cartier (see [17]). Since the Cartier operator

is also the trace map in Grothendieck duality theory for the finite flat map F , we have a perfect pairing

$$(2.2.2) \quad F_*(\Omega_X^j) \otimes F_*(\Omega_X^{n-j}) \rightarrow \Omega_X^n$$

where $n = \dim(X)$, and the pairing is given by $(\omega_1, \omega_2) \mapsto C(w_1 \wedge w_2)$. This pairing is perfect and \mathcal{O}_X -bilinear (see [32]).

In particular, on applying $\text{Hom}(-, \Omega_X^n)$ to the exact sequence

$$(2.2.3) \quad 0 \rightarrow B_X^n \rightarrow Z_X^n \rightarrow \Omega_X^n \rightarrow 0$$

we get

$$(2.2.4) \quad 0 \rightarrow \mathcal{O}_X \rightarrow F_*(\mathcal{O}_X) \rightarrow B_X^1 \rightarrow 0$$

2.3. de Rham-Witt cohomology. The standard reference for de Rham-Witt cohomology is [17]. Throughout this section, the following notations will be in force. Let k be an algebraically closed field of characteristic $p > 0$, and X a smooth, projective variety over k . Let $W = W(k)$ be the ring of Witt vectors of k . Let $K = W[1/p]$ be the quotient field of W . Note that as k is perfect, W is a Noetherian local ring with a discrete valuation and with residue field k . For any $n \geq 1$, let $W_n = W(k)/p^n$. W comes equipped with a lift $\sigma : W \rightarrow W$, of the Frobenius morphism of k , which will be called the Frobenius of W . We define a non-commutative ring $R^0 = W_\sigma[V, F]$, where F, V are two indeterminate subject to the relations $FV = VF = p$ and $Fa = \sigma(a)F$ and $aV = V\sigma(a)$. The ring R^0 is called the Dieudonne ring of k . The notation is borrowed from [20].

Let $\{W_n\Omega_X^*\}_{n \geq 1}$ be the de Rham-Witt pro-complex constructed in [17]. It is standard that for each $n \geq 1, i, j \geq 0$, $H^i(X, W_n\Omega_X^j)$ are of finite type over W_n . We define

$$(2.3.1) \quad H^i(X, W\Omega_X^j) = \varprojlim_n H^i(X, W_n\Omega_X^j),$$

which are W -modules of finite type up to torsion. These cohomology groups are called Hodge-Witt cohomology groups of X .

Definition 2.3.1. X is Hodge-Witt if for $i, j \geq 0$, the Hodge-Witt cohomology groups $H^i(X, W\Omega_X^j)$ are finite type W -modules.

The properties of the de Rham-Witt pro-complex are reflected in these cohomology modules and in particular we note that for each i, j , the Hodge-Witt groups $H^i(X, W\Omega_X^j)$ are left modules over R^0 . The complex $W\Omega_X^*$ defined in a natural manner from the de Rham-Witt pro-complex computes the crystalline cohomology of X and in particular there is a spectral sequence

$$(2.3.2) \quad E_1^{i,j} = H^i(X, W\Omega_X^j) \Rightarrow H_{\text{cris}}^*(X/W)$$

This spectral sequence induces a filtration on the crystalline cohomology of X which is called the slope filtration and the spectral sequence above is called the slope spectral sequence (see [17]). It is standard (see [17] and [20]) that the slope spectral sequence degenerates at E_1 modulo torsion (i.e. the differentials are zero on tensoring with K) and at E_2 up to finite length (i.e. all the differential have images which are of finite length over W).

In dealing with the slope spectral sequence it is more convenient to work with a bigger ring than R^0 . This ring was introduced in [20]. Let $R = R^0 \oplus R^1$ be a graded W -algebra which is generated in degree 0 by variables F, V with the properties listed earlier (so R^0 is the Dieudonne ring of k) and R^1 is a bimodule over R^0 generated in degree 1 by d with the

properties $d^2 = 0$ and $FdV = d$, and $da = ad$ for any $a \in W$. The algebra R is called the Raynaud-Dieudonné ring of k (see [20]). The complex $(E_1^{*,i}, d_1)$ is a graded module over R and is in fact a coherent, left R -module (in a suitable sense, see [20]).

For a general variety X , the de Rham-Witt cohomology groups are not of finite type over W , and the structure of these groups reflects the arithmetical properties of X . For instance, in [5], [20] it is shown that for ordinary varieties $H^i(X, W\Omega^j)$ are of finite type over W .

3. PRIMES OF ORDINARY REDUCTION FOR $K3$ SURFACES

3.1. Serre's conjecture. The following more general question, which is one of the motivating questions for this paper, is the following conjecture which is well-known and was raised initially for abelian varieties by Serre:

Conjecture 3.1.1. Let X/K be a smooth projective variety over a number field. Then there is a positive density of primes v of K for which X has good ordinary reduction at v .

Let K be a number field, and let X denote either an abelian variety or a $K3$ surface defined over K . Our aim in this section is to show that there is a finite extension L/K of number fields, such that the set of primes of L at which X has ordinary reduction in the case of $K3$ surfaces, or has p -rank at least two if X is an abelian variety of dimension at least two, is of density one. Our proof closely follows the method of Ogus for abelian surfaces (see [40, page 372]).

We note here that a proof of the result for a class of $K3$ surfaces was also given by Tankeev (see [46]) under some what restrictive hypothesis. The question of primes of ordinary reduction for abelian varieties has also been treated recently by R. Noot (see [37]), R. Pink (see [41]) and more recently A. Vasiu (see [48]) has studied the question for a wider class of varieties. The approach adopted by these authors is through the study of Mumford-Tate groups.

Let \mathcal{O}_K be the ring of integers of K ; for a finite place v of K lying above a rational prime p , let \mathcal{O}_v be the completion of \mathcal{O}_K with respect to v and let k_v be the residue field at v of cardinality $q_v = p^{c_v}$. Assume that v is a place of good reduction for X as above and write X_v for the reduction of X at v . We recall here the following facts:

3.2. Trace of Frobenius. The Frobenius endomorphism F_v is an endomorphism of the l -adic cohomology groups $H_l^i := H_{et}^i(X \otimes \overline{K}, \mathbb{Q}_l)$ for a prime $l \neq p$. The l -adic characteristic polynomial $P_{i,v}(t) = \det(1 - tF_v \mid H_{et}^i(X \otimes \overline{K}, \mathbb{Q}_l))$ is an integral polynomial and is independent of l . Let

$$a_v = \text{Tr}(F_v \mid H_{et}^2(X \otimes \overline{K}, \mathbb{Q}_l))$$

denote the trace of the l -adic Frobenius acting on the second étale cohomology group. Then a_v is a rational integer (see [40]).

3.3. Deligne Weil estimate. (Deligne-Weil estimates) [10]: It follows from Weil estimates proved by Weil for abelian varieties and by Deligne in general that

$$|a_v| \leq dp$$

where $d = \dim H_l^2$ is a constant independent of the place v .

3.4. Katz-Messing theorem. Let ϕ_v denote the crystalline Frobenius on $H_{\text{cris}}^i(X/W(k_v))$. $\phi_v^{c_v}$ is linear over $W(k_v)$, and the characteristic polynomials of the crystalline Frobenius and the l -adic Frobenius are equal:

$$P_{i,v}(t) = \det(1 - t\phi_v^{c_v} \mid H_{\text{cris}}^i(X/W(k_v)) \otimes K_v),$$

(see [25]).

3.5. Semi-simplicity of the crystalline Frobenius. If X is a $K3$ -surface, then it is known by [40, Theorem 7.5] that the crystalline Frobenius $\phi_v^{c_v}$ is semi-simple. We remark that although this result is not essential in the proof of the theorem, it simplifies the proof.

3.6. Mazur's theorem. We recall from [30], [11], [4] the following theorem of B. Mazur. There are two parts to the theorem of Mazur that we require and we record them separately for convenience. After inverting finitely many primes $v \in S$ in K , we can assume that X has good reduction outside S . Using Proposition 3.7.1 (see below) we can assume that $H^i(X, \Omega_X^j)$ and $H^i(X/W(k_v))$ are torsion-free outside a finite set of primes of K . As X is defined over characteristic zero, the Hodge to de Rham spectral sequence degenerates at E_1 stage. Thus all the hypothesis of Mazur's theorem are satisfied. The two parts of Mazur's theorem that we require are the following:

3.6.1. Hodge and Newton polygons: Mazur's proof of Katz's conjecture. Let L_v be a finite extension of field of fractions of $W(k_v)$, over which the polynomial $P_{i,v}(t)$ splits into linear factors. Let w denote a valuation on L_v , such that $w(p) = 1$. The Newton polygon of the polynomial $P_{i,v}(t)$ lies above the Hodge polygon in degree i , defined by the Hodge numbers $h^{j,i-j}$ of degree i . Moreover they have the same endpoints.

3.6.2. Divisibility. The crystalline Frobenius ϕ_v is divisible by p^i when restricted to

$$F^i H_{\text{dR}}^j(X/W(k_v)) := \mathbb{H}(X/W(k_v), \Omega_X^{\geq i}).$$

3.7. Crystalline torsion. We will also need the following proposition (from [21]) which is certainly well-known but as we use it in the sequel, we record it here for convenience.

Proposition 3.7.1. *Let X/K be a smooth projective variety. Then for all but finitely many nonarchimedean places v , the crystalline cohomology $H_{\text{cris}}^i(X_v/W(k_v))$ is torsion free for all i .*

Proof. We choose a smooth model $\mathcal{X} \rightarrow \text{Spec}(\mathcal{O}_K) - V(I)$ for some non-zero proper ideal $I \subset \mathcal{O}_K$. The relative de Rham cohomology of the smooth model \mathcal{X} is a finitely generated \mathcal{O}_K -module and has bounded torsion. After inverting a finite set S of primes, we can assume that $H_{\text{dR}}^i(\mathcal{X}, \mathcal{O}_S)$ is a torsion-free \mathcal{O}_S module, where \mathcal{O}_S is the ring of S -integers in K . By the comparison theorem of Berthelot (see [3]), there is a natural isomorphism of the crystalline cohomology of X_p to that of the de Rham cohomology of the generic fiber of a lifting to \mathbb{Z}_p .

$$H_{\text{cris}}^i(X_v/W(k_v)) \cong H_{\text{dR}}^i(\mathcal{X}, \mathcal{O}_S) \otimes W(k_v).$$

This proves our proposition. Moreover the proof shows that we can assume after inverting some more primes, that the Hodge filtrations $F^j H_{\text{dR}}^i(\mathcal{X}, \mathcal{O}_S)$ are also locally free over \mathcal{O}_S , such that the sub-quotients are also locally free modules. \square

We first note the following lemma which is fundamental to the proof.

Lemma 3.7.2 (Joshi-Rajan). *With notation as above, assume the following:*

- (1) if X is a $K3$ -surface, then X does not have ordinary reduction at v .
- (2) if X is an abelian variety, then the p -rank of the reduction of X at v , is at most 1.

Then $p|a_v$.

Proof. Let w be a valuation as in 3.6.1 above. If X is an abelian variety defined over the finite field k_v , then the p -rank of X is precisely the number of eigenvalues of the correct power of the crystalline Frobenius acting on $H^1(X/W(k_v)) \otimes \mathbb{Q}_p$, which are p -adic units. Suppose now α is an eigenvalue of the crystalline Frobenius $\phi_v^{c_v}$ acting on $H^2(X/W(k_v)) \otimes \mathbb{Q}_p$. In case (2), the hypothesis implies that $w(\alpha)$ is positive, and hence $w(a_v)$ is strictly positive. As a_v is a rational integer, the lemma follows.

When X is a $K3$ -surface, it follows from the shapes of the Newton and Hodge polygons, that ordinarity is equivalent to the fact that precisely one eigenvalue of $\phi_v^{a_v}$ acting on $H^2(X/W(k_v)) \otimes \mathbb{Q}_p$ is a p -adic unit. Hence if X is not ordinary, then for any α as above, we have $w(\alpha_v)$ is positive. Again since a_v is a rational integer, the lemma follows. \square

The following theorem was proved in [21]. We note that the first assertion of Theorem 3.7.3 was also independently established by Fedor Bogomolov and Yuri Zarhin (see [6]). Their method is different method from the one adopted here.

Theorem 3.7.3 (Joshi-Rajan). *Let X be a $K3$ surface or an abelian variety of dimension at least two defined over a number field K . Then there is a finite extension L/K of number fields, such that*

- (1) if X is a $K3$ -surface, then $X \times_K L$ has ordinary reduction at a set of primes of density one in L .
- (2) if X is an abelian variety, then there is set of primes O of density one in L , such that the reduction of $X \times_K L$ at a prime $p \in O$ has p -rank at least two.

Proof. Our proof follows closely the method of Serre and Ogus (see [40]). Fix a prime l , and let ρ_l denote the corresponding galois representation on H_l^2 . The galois group G_K leaves a lattice V_l fixed, and let $\bar{\rho}_l$ denote the representation of G_L on $V_l \otimes \mathbb{Z}/l\mathbb{Z}$. Let L be a galois extension of \mathbb{Q} , containing K and the l^{th} roots of unity, and such that for $\sigma \in G_L$, $\bar{\rho}_l(\sigma) = 1$. We have

$$a_v \equiv d \pmod{l},$$

where $d = \dim H_l^2$.

Let v be a prime of L of degree 1 over \mathbb{Q} , lying over the rational prime p . Since p splits completely in L , and L contains the l^{th} roots of unity, we have $p \equiv 1 \pmod{l}$. Now choose $l > d$. Since $|a_v| \leq dp$ and is a rational integer divisible by p from the above lemma, it follows on taking congruences modulo l that

$$a_v = \pm dp.$$

Now a_v is the sum of d algebraic integers each of which is of absolute value p with respect to any embedding. It follows that all these eigenvalues must be equal, and equals $\pm p$. Hence we have that

$$F_v = \pm pI$$

as an operator on H_l^2 . By the semi-simplicity of the crystalline Frobenius for abelian varieties and $K3$ -surfaces [40], it follows that $\phi_v = \pm pI$. But this contradicts the divisibility property of the crystalline Frobenius 3.6.2, that the crystalline Frobenius is divisible by p^2

on $F^2 H_{dR}^2(X \times k_v)$. Hence v has to be a prime of ordinary reduction, and this completes the proof of our theorem. \square

Ogus' method can in fact be axiomatized to give positive density results whenever certain cohomological conditions are satisfied. We present this formulation for the sake of completeness.

Proposition 3.7.4 (Joshi-Rajan). *Let X be a smooth projective variety over a number field K . Assume the following conditions are satisfied:*

- (1) $\dim H^2(X, \mathcal{O}_X) = 1$,
- (2) *The action of the crystalline Frobenius of the reduction $X_{\mathfrak{p}}$ of X at a prime \mathfrak{p} is semi-simple for all but finite number of primes \mathfrak{p} of K .*

Then the galois representation $H_{et}^2(X, \mathbb{Q}_{\ell})$ is ordinary at a set of primes of positive density in K and the F -crystal $H_{cris}^2(X_{\mathfrak{p}}/W)$ is ordinary for these primes. In other words, the motive $H^2(X)$ has ordinary reduction for a positive density of primes of K .

4. PRIMES OF HODGE-WITT REDUCTION

4.1. Hodge-Witt reduction. Let X be a smooth projective variety over a number field K . We fix a model $\mathcal{X} \rightarrow \text{Spec}(\mathcal{O}_K)$ which is regular, proper and flat and which is smooth over a suitable non-empty subset of $\text{Spec}(\mathcal{O}_K)$. All our results are independent of the choice of the model. In what follows we will be interested in the smooth fibers of the map $\mathcal{X} \rightarrow \text{Spec}(\mathcal{O}_K)$, in other words we will always consider primes of good reduction. Henceforth p will always denote such a prime and the fiber over this prime will be denoted by X_p .

Since ordinary varieties are Hodge-Witt, we can formulate a weaker version of Conjecture 3.1.1.

Conjecture 4.1.1 (Joshi-Rajan). *Let X/K be a smooth projective variety over a number field then X has Hodge-Witt reduction modulo a set of primes of K of positive density.*

Remark 4.1.2. Let us understand what our conjecture means in the context of a K3 surface. Let X/K be a smooth, projective K3 surface over a number field K . So Conjecture 4.1.1 predicts in this case that there are infinitely many primes of Hodge-Witt reduction for X . Now X has Hodge-Witt reduction at a prime \mathfrak{p} of K if and only if X is of finite height (i.e. the formal group attached to X by [1] is of finite height).

As we remarked in the Introduction, Conjecture 4.1.1 and Conjecture 3.1.1 are not equivalent (see Example 6.1.1). The two however are related and the following clarifies the relationship between Conjecture 3.1.1 and Conjecture 4.1.1. Our theorem is the following:

Theorem 4.1.3. *Let X be a smooth, projective variety over a number field K . Then the following are equivalent:*

- (1) *There are infinitely many primes of ordinary reduction for X .*
- (2) *There are infinitely many primes of ordinary reduction for $X \times_K X$*
- (3) *There are infinitely many primes of Hodge-Witt reduction for $X \times_K X$.*

Proof. Let us equip ourselves with flat, regular, proper models for X and $X \times_K X$ over a Zariski-open subscheme of $U \subset \text{Spec}(\mathcal{O}_K)$ with smooth, proper, fibres over U . Let $\mathfrak{p} \in U$ be a prime of good, ordinary reduction for X . As a product of ordinary varieties is ordinary (see [19]), so $X \times_K X$ is also ordinary at \mathfrak{p} . Thus (1) \implies (2). Since any prime \mathfrak{p} of good,

ordinary reduction for $X \times_K X$ is also a prime of good, Hodge-Witt reduction for $X \times_K X$, so we have (2) \implies (3). Let us prove that (3) \implies (1). Assume \mathfrak{p} is a prime of Hodge-Witt reduction for $X \times_K X$. By [12, III, Prop 2.1(ii) and Prop 7.2(ii)] we see that a product is Hodge-Witt if and only if at least one factor is ordinary and other is Hodge-Witt. Thus if $X \times_K X$ is Hodge-Witt at \mathfrak{p} then X must be ordinary at \mathfrak{p} . Thus X is ordinary at \mathfrak{p} . \square

4.2. Hodge-Witt Abelian varieties. Let X be a p -divisible group over a perfect field k of characteristic $p > 0$. Let X^t denote the Cartier dual of X and let $X_{(0,1)}$ be the local-local part of X . Following [8, Def 3.4.2] we say that a p -divisible group X is of *extended Lubin-Tate type* if $\dim X_{(0,1)} \leq 1$ or $\dim X_{(0,1)}^t \leq 1$. Equivalently X is of extended Lubin-Tate type if X is isogenous to an ordinary p -divisible group or X is isogenous to a product of an ordinary p -divisible group and one copy of the p -divisible group of a supersingular elliptic curve. We begin with the following proposition:

Proposition 4.2.1. *Let k be a perfect field of characteristic $p > 0$. Let A be an abelian variety over k . Then the following are equivalent:*

- (1) *A is Hodge-Witt.*
- (2) *A has p -rank $\geq \dim(A) - 1$.*
- (3) *The slopes of Frobenius of A are in $\{0, 1, \frac{1}{2}\}$ and multiplicity of $\frac{1}{2}$ is at most two.*
- (4) *The p -divisible group $A[p^\infty]$ is of extended Lubin-Tate type.*

Proof. The equivalence (1) \iff (2) is [18]. The equivalence (2) \iff (3) \iff (4) is immediate from the definition. \square

Remark 4.2.2. In [8] an abelian variety A with p -rank equal to $\dim(A) - 1$ is called an *almost ordinary abelian variety*. Thus we see that A is Hodge-Witt if and only if A is ordinary or A is almost ordinary.

Remark 4.2.3. Properties of Hodge-Witt varieties and non-Hodge-Witt varieties can be rather distinct. A classic example of this phenomenon is implicit in [8]. Since the phrase *Hodge-Witt* is never mentioned in [8] we provide the following transliteration of the relevant results.

Theorem 4.2.4 (Reformulation of [8, Theorem 3.5.1, Corollary 3.5.6 and Remark 3.5.7]). *Let k be a perfect field of characteristic $p > 0$, let A be an abelian variety with a CM algebra K with $\dim(K) = 2 \dim(A)$ and an embedding $K \hookrightarrow \text{End}_k^0(A) = \text{End}_k^0(A)$. If A is Hodge-Witt, then there exists an isogeny $A' \rightarrow A$ such that A' admits a CM-lifting to characteristic zero. On the other hand if A is not Hodge-Witt, there exists an isogeny $A' \rightarrow A$, such that A' does not admit any CM-lifting to characteristic zero.*

4.3. Primes of Hodge-Witt reduction for abelian threefolds. Our next theorem shows that the Conjecture 4.1.1 is true for abelian varieties of dimension at most three. This result is implicit in [21].

Theorem 4.3.1 (Joshi-Rajan). *Let A/K be an abelian variety of dimension at most three over a number field K . Then there exists a finite extension L/K and a set of primes of density one in L at which A has Hodge-Witt reduction.*

Proof. The case $\dim(A) = 1$ is immediate as smooth, projective curves are always Hodge-Witt. Next assume that $\dim(A) = 2$, then we know by Ogus's result that an abelian surface

defined over a number field has a positive density of primes of ordinary reduction. So we are done in this case.

Thus we need to address $\dim(A) = 3$. Recall that an abelian variety A over a perfect field is Hodge-Witt if and only if the p -rank of A is at least $\dim(A) - 1$ (see [18]). This together with Theorem 3.7.3 gives the result. \square

For surfaces the geometric genus appears to detect the size of the set of primes which is predicted in Conjecture 4.1.1.

Theorem 4.3.2 (Joshi-Rajan). *Let X be a smooth, projective surface with $p_g(X) = 0$, defined over a number field K . Then for all but finitely many primes \mathfrak{p} , X has Hodge-Witt reduction at \mathfrak{p} .*

Proof. By the results of [38], [17], [20], it suffices to verify that

$$H^2(X_{\mathfrak{p}}, W(\mathcal{O}_{X_{\mathfrak{p}}})) = 0$$

for all but a finite number of primes \mathfrak{p} of K . But the assumption that $p_g(X) = 0$ entails that $H^2(X, \mathcal{O}_X) = H^0(X, K_X) = 0$. Hence by the semicontinuity theorem, for all but finite number of primes \mathfrak{p} of K , the reduction $X_{\mathfrak{p}}$ also has $p_g(X_{\mathfrak{p}}) = 0$. Then by [17], [38] one sees that $H^2(X_{\mathfrak{p}}, W(\mathcal{O}_{X_{\mathfrak{p}}})) = 0$ and so $X_{\mathfrak{p}}$ is Hodge-Witt at any such prime. \square

Corollary 4.3.3 (Joshi-Rajan). *Let X/K be an Enriques surface over a number field K . Then X has Hodge-Witt reduction modulo all but finite number of primes of K .*

Proof. This is immediate from the fact that for an Enriques surface over K , $p_g(X) = 0$. \square

When X is a smooth Fano surface over a number field, one can prove a little more:

Theorem 4.3.4 (Joshi-Rajan). *Let X be a smooth, projective Fano surface, defined over a number field K . Then for all but finitely many primes p , X has ordinary reduction at p and moreover the de Rham-Witt cohomology of X_p is torsion free.*

Proof. It follows from the results of [34] and [16], that if X is a smooth, projective and Fano variety X over a number field, for all but finitely many primes p , the reduction X_p is Frobenius split. By [22, Proposition 3.1] we see the reduction modulo all but finitely many primes p of K gives an ordinary surface. Then by Lemma 9.5 of [5] and Proposition 3.7.1, the result follows. \square

Example 4.3.5 (Joshi-Rajan). Let $K = \mathbb{Q}$ and $X \subset \mathbb{P}^n$ be any Fermat hypersurface of degree m and $n \geq 6$. If $m < n + 1$ then this hypersurface is Fano but by [47] this hypersurface does not have Hodge-Witt reduction at primes p satisfying $p \not\equiv 1 \pmod{m}$. This gives examples of Fano varieties which are (F -split but are) not Hodge-Witt or ordinary.

Remark 4.3.6 (Joshi-Rajan). It is clear from Example 4.3.5 that there exist Fano varieties over number fields which have non-Hodge-Witt reduction modulo an infinite set of primes and thus this indicates that in higher dimension $p_g(X)$ is not a good invariant for measuring this behavior. The following question and subsequent examples suggests that the Hodge level may intervene in higher dimensions (see [9, 42] for the definition of Hodge level).

Question 4.3.7 (Joshi-Rajan). Let X/K be a smooth, projective Fano variety over a number field. Assume that X has Hodge level ≤ 1 in the sense of [9, 42]. Then does X have Hodge-Witt reduction modulo all but a finite number of primes of K ?

Remark 4.3.8 (Joshi-Rajan). A list of all the smooth complete intersection in \mathbb{P}^n which are of Hodge level ≤ 1 is given in [42] and one knows from [45] that complete intersections of Hodge level 1 are Hodge-Witt.

Remark 4.3.9. In [23] we have answered the Question 4.3.7 affirmatively for $\dim(X) = 3$ where the Hodge level condition is automatic.

4.4. Hodge-Witt reduction of CM abelian varieties. We will recall a few facts about CM abelian varieties. For a modern reference for CM abelian varieties reader may consult [8]. Let K be a CM field. Let L be a number field and let A be an abelian variety over L with $\text{End}_L^0(A) = K$. In other words L is an abelian variety with complex multiplication (“CM”) by K . We will assume that , by passing to a finite extension i if required that L contains

- (1) the galois closure N of K
- (2) and that A has good reduction at all finite primes of L .

Assumption (2) is guaranteed by the theory of Complex multiplication (see [8]). Let Φ be the CM-type of A . Let $E(K, \Phi)$ be the reflex field. In this situation by [49] we know that the Dieudonne module of the reduction \bar{A} at any finite prime \mathfrak{p} of L is determined by Φ and the splitting of rational primes in K . In particular it is possible to determine completely the reduction type of A from the decomposition of p in \mathcal{O}_K . But in general this is a complicated combinatorial problem (see [14, 44, 50]) depending on the galois closure of K and also on the CM-type. In the present section we give sufficient conditions on prime decomposition of p which allows us to conclude the infinitude of primes of Hodge-Witt and ordinary reduction for A . Before proceeding we make some elementary observations.

Definition 4.4.1. Let K/\mathbb{Q} be a finite extension. Let p be a prime number which is unramified in K . We say that p *splits almost completely* in K and we have a prime factorization:

$$(p) = \prod_{i=1}^n \mathfrak{p}_i,$$

with $n \geq [K : \mathbb{Q}] - 1$ if $[K : \mathbb{Q}] > 2$ and $n = [K : \mathbb{Q}]$ if $[K : \mathbb{Q}] \leq 2$. We say that p *splits completely* in K if we have a prime factorization:

$$(p) = \prod_{i=1}^n \mathfrak{p}_i,$$

with $n = [K : \mathbb{Q}]$. Let f_i be the degree of the residue field extension of \mathfrak{p}_i over \mathbb{Z}/p . Then the *residue field degree sequence* of p in K is the tuple (f_1, f_2, \dots) .

It is clear from Definition 4.4.1 that if p splits completely in K then p splits almost completely in K . Also from our definition p splits almost completely in a quadratic extension if and only if p splits completely. The following proposition is an elementary consequence of Definition 4.4.1 and standard facts about splitting of primes in K but we give a proof for completeness.

Proposition 4.4.2. *Suppose $[K : \mathbb{Q}] = 2g$ be a CM field and let $K_0 \subset K$ be the totally real subfield of K . Then we have the following assertions:*

- (1) *If p splits completely in K then p splits completely in K_0 .*

- (2) If p splits almost completely, but not completely in K then p splits completely in K_0 and exactly one prime lying over p in K_0 is inert in K and the rest split in K .
- (3) If K/\mathbb{Q} is galois then p splits almost completely in K if and only if p splits completely in K .

Proof. Let $[K : \mathbb{Q}] = 2g$. Let

$$(4.4.1) \quad (p) = \mathfrak{p}_1 \cdots \mathfrak{p}_n$$

be the prime factorization of p in K (recall that by definition p splits completely means p is unramified) and let

$$(4.4.2) \quad (p) = \mathfrak{q}_1 \cdots \mathfrak{q}_\ell,$$

be the prime factorization of p in K_0 . Let m be the number of primes in $\mathfrak{q}_1, \dots, \mathfrak{q}_\ell$ which split in K . By renumbering the $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ we can assume that $\mathfrak{p}_1, \dots, \mathfrak{p}_{2m}$ are the primes lying over the split primes $\mathfrak{q}_1, \dots, \mathfrak{q}_m$. The rest lie over primes of K_0 (over p) which are inert in K . Thus we have

$$(4.4.3) \quad 2g = 2m + (\ell - m)$$

equivalently

$$(4.4.4) \quad 2g = \ell + m.$$

Further if $f(\mathfrak{p}|p)$ denotes the degree of the residue field of a prime \mathfrak{p} lying over p then we have for K_0 the equation

$$(4.4.5) \quad \ell \leq \sum_{i=1}^{\ell} f(\mathfrak{q}_i|p) = g.$$

Thus we see that $2g = \ell + m \leq 2\ell \leq 2g$ as $m \leq \ell$. Thus we have $2g \leq 2\ell \leq 2g$ so $\ell = g$. Thus p splits completely in K_0 . This proves the first assertion.

Now suppose p splits almost completely in K but not completely in K . Then $[K : \mathbb{Q}] = 2g > 2$ by definition (see 4.4.1) as p splits almost completely for $[K : \mathbb{Q}] = 2$ if and only if p splits in K . So if p is a such a prime then $n = 2g - 1 > 1$ which gives $g > 1$. Further, using the notation established in the previous case we have

$$(4.4.6) \quad 2m + (\ell - m) = \ell + m = 2g - 1.$$

Thus $\ell + m = 2g - 1$. We claim that $m \geq g - 1$. Suppose $m < g - 1$ then

$$(4.4.7) \quad 2g - 1 = \ell + m < \ell + g - 1,$$

so $g < \ell$. But as $\ell \leq \sum_{i=1}^{\ell} f(\mathfrak{q}_i|p) = g$ one gets $\ell \leq g$. So $g < \ell$ and $\ell \leq g$ which is a contradiction. This gives $m \geq g - 1$. If $m = g$ then we are in the completely split case. So $m = g - 1$. Then $2g - 1 = \ell + m = \ell + g - 1$ and hence $\ell = g$. Further as $g > 1$ so $m = g - 1 > 1$ and hence exactly one prime of $\mathfrak{q}_1, \dots, \mathfrak{q}_g$ is inert in K as claimed. This proves the second assertion.

Finally suppose K/\mathbb{Q} is galois and p splits almost completely then p is completely split as all the residue field degrees are equal, while from the preceding discussion exactly one residue field degree of prime lying over p in K is two if p splits almost completely but not completely split, so we are done by the second assertion. \square

The following is also immediate from this proof.

Corollary 4.4.3. *Let K be a CM field. Let p be a rational prime which is unramified in K . Then the following are equivalent:*

- (1) p splits almost completely in K .
- (2) the residue field degree sequence of p in K is $(1, 1, 1, \dots)$ or $(2, 1, 1, 1, \dots)$.

We are now ready to prove our theorem.

Theorem 4.4.4. *Let A, L, K be as above with K a CM field. Let \mathfrak{p}_L be a prime of L lying over an unramified prime p of \mathbb{Q} . If p splits almost completely in K , then A has Hodge-Witt reduction at \mathfrak{p}_L .*

Proof. We follow a method due to [14, 44, 50]—especially [44] who proved the result when p is completely split (in which case A has ordinary reduction at \mathfrak{p}_L)—also see [14, 50]. We note that [14] spells out the details quite well (for abelian surfaces—and is also enough to prove what we need here). Suppose \mathfrak{p}_L is a prime lying over p in L . Let κ be the residue field of \mathfrak{p}_L . Then the reduction of A at \mathfrak{p}_L is an abelian scheme over κ . It will be convenient to extend the base field of the reduction from κ to an algebraic closure $\bar{\kappa}$ of κ . Let X be the p -divisible group of this abelian scheme over $\bar{\kappa}$. Let \mathcal{O}_K be the ring of integers of K , let $K_0 \subset K$ be the totally real subfield and \mathcal{O}_{K_0} be the ring of integers of K_0 . Let $\sigma : \mathcal{O}_K \rightarrow \mathcal{O}_K$ be complex conjugation. Then \mathcal{O}_K operates on the p -divisible group X .

Thus we are in the following situation: we have a p -divisible group X over $\bar{\kappa}$ with CM by K and we have to show that if p splits almost completely in K then X is of extended Lubin-Tate type.

Let $M = M(X)$ be the Dieudonné module of X . Then M is a $W = W(\bar{\kappa})$ -module and let us note, by the standard theory of complex multiplication (see [8]), that M is an \mathcal{O}_K -module of rank one and hence, or at any rate, M is an $\mathcal{O}_K \otimes \mathbb{Z}_p$ -module. We may decompose $\mathcal{O}_K \otimes \mathbb{Z}_p$ using the factorization of (p) . This also gives us a decomposition of the Dieudonné module $M = M(X)$ of X .

Assume that $[K : \mathbb{Q}] = 2g$ and $(p) = \prod_{i=1}^n \mathfrak{p}_i$. Then

$$\mathcal{O}_K \otimes \mathbb{Z}_p = \prod_{i=1}^n W(k(\mathfrak{p}_i))$$

where $k(\mathfrak{p}_i)$ is the residue field of \mathfrak{p}_i . Then we have to show that if $n \geq 2g - 1$ then X is of extended Lubin-Tate type.

If $n = 2g$ then p splits completely and we are done by [44], but we recall his argument here as it is also needed in case p splits almost completely. If p is completely split, then by Proposition 4.4.2(1) every \mathfrak{p}_i pairs uniquely with a prime \mathfrak{p}_j such that $\mathfrak{p}_j = \sigma(\mathfrak{p}_i)$ (i.e. $\mathfrak{p}_i, \mathfrak{p}_j$ live over a prime of \mathcal{O}_{K_0} which splits completely in \mathcal{O}_K). The factor of M , corresponding to $\mathfrak{p}_i, \mathfrak{p}_j$ provides a Dieudonné module of a p -divisible group, which descends to the residue field of the unique prime of \mathcal{O}_{K_0} which splits into $\mathfrak{p}_i, \mathfrak{p}_j$ in \mathcal{O}_K , gives a p -divisible group of type $G_{1,0} \times G_{0,1}$ (in the standard notation of p -divisible groups) with slopes 0, 1. If p is completely split, then there are exactly g such factors and hence X is ordinary p -divisible group and hence A has ordinary reduction at \mathfrak{p}_L . If p splits almost completely but does not split completely then by Proposition 4.4.2(2) all but one of \mathfrak{p}_i can be paired with exactly one \mathfrak{p}_j as above and there are exactly $2g - 2$ such primes, while the remaining prime \mathfrak{p}_{2g-1} (after renumbering if required), lives over the unique prime of \mathcal{O}_{K_0} lying over p which is inert in K . As is shown in [44], the factor of M corresponding to \mathfrak{p}_{2g-1} gives an indecomposable Dieudonné module of rank two over the residue field of \mathfrak{p}_{2g-1} with slope $\frac{1}{2}$, and hence a

p -divisible group $G_{1,1}$. Thus in this case X is extended Lubin-Tate group of height $2g$. This completes the proof of the theorem. \square

Theorem 4.4.5. *Let A, L, K be as in 4.4. Then there exists infinitely many primes of Hodge-Witt reduction. In particular Conjecture 4.1.1 is true for abelian varieties with complex multiplication and in general the density of such primes is greater than the density of primes of ordinary reduction.*

Proof. This is a consequence of Theorem 4.4.4 and the Chebotarev density theorem. Let N be the galois closure of K in $\bar{\mathbb{Q}}$, let $G = \text{Gal}(N/\mathbb{Q})$. Let $H \subset G$ be the subgroup fixing H . Then action of G on the coset space G/H embeds $G \hookrightarrow S_n$ where $n = [G : H]$ and in particular $\text{Frob}_p \in G$ acts on G/H (by permutation). This action depends only on the conjugacy class of Frob_p . It is well-known, see for instance [28] and [27] that the splitting type is determined by the cycle decomposition of Frob_p on G/H and that the splitting type of p in K is determined by the sequence of residue field degrees. In particular we can “read off” all the information we need from the table of conjugacy classes of G . By Corollary 4.4.3 we see that p splits completely if and only if the cycle decomposition is identity, and the p splits almost completely if the cycle decomposition is identity or exactly one transposition. Let G_{tr} be the union of the conjugacy classes of these two types. Then $G_{tr} \supseteq \{1\}$ and hence $G_{tr} \neq \emptyset$. In particular we see that the density of primes of Hodge-Witt reduction is $\frac{|G_{tr}|}{|G|} \geq \frac{1}{|G|} > 0$ and this proves the theorem. To prove the last assertion it is enough to give an example. See Example 6.1.3 below. \square

Let us first state some obvious corollaries.

Corollary 4.4.6. *Let K be a CM field with galois closure N/\mathbb{Q} and $G = \text{Gal}(N/\mathbb{Q})$. Let G_{tr} be as above. Then the density of primes of Hodge-Witt reduction for A is at least $\frac{|G_{tr}|}{|G|}$ and the density of primes of ordinary reduction is at least $\frac{1}{|G|}$.*

Corollary 4.4.7. *If A, L, K, \mathfrak{p}_L are as in Theorem 4.4.4. Then Conjecture 3.1.1 is true for A .*

Proof. This is clear from the preceding results. \square

Corollary 4.4.8. *If A, L, K, \mathfrak{p}_L are as above, then Question 5.1.2 has an affirmative answer for CM abelian varieties.*

Corollary 4.4.9. *Let A/L be an abelian variety over a field of characteristic zero. Assume A has complex multiplication by a CM field K . Then the conjecture of Mustata-Srinivas (see [35, Conjecture 1.1]) is true for A .*

5. NON HODGE-WITT REDUCTION

Let us briefly discuss non Hodge-Witt reduction. By definition a smooth, projective variety is non Hodge-Witt if and only if some de Rham-Witt cohomology group $H^i(X, W\Omega_X^j)$ is not finitely generated as a W -module. In other words, for some $i, j \geq 0$, $H^i(X, W\Omega_X^j)$ has infinite torsion. In this section we briefly describe what we know about primes of non Hodge-Witt reduction and what we expect to be true and how this relates to some of other well-known conjectures.

5.1. Hodge-Witt torsion. We include here some observations probably well-known to the experts, but we have not found them in print. We assume as in the previous section that X/K is smooth projective variety over a number field and that we have fixed a regular, proper model smooth over some open subscheme of the ring of integers of K and whose generic fiber is X .

Before we proceed we record the following:

Proposition 5.1.1 (Joshi-Rajan). *Let X/K be a smooth projective variety over a number field K . Then there exists an integer N such that for all primes \mathfrak{p} in K lying over any rational prime $p \geq N$, the following dichotomy holds*

- (1) *either for all $i, j \geq 0$, the Hodge-Witt groups are free W -modules (of finite type), or*
- (2) *there is some pair i, j such that $H^i(X_{\mathfrak{p}}, W\Omega_{X_{\mathfrak{p}}}^j)$ has infinite torsion.*

Proof. Choose a finite set of primes S of K , such that X has a proper, regular model over $\text{Spec}(\mathcal{O}_{K,S})$, where $\mathcal{O}_{K,S}$ denotes the ring of S -integers in K . Choose N large enough so that for any prime \mathfrak{p} lying over a rational prime $p > N$, we have $\mathfrak{p} \notin S$, and all the crystalline cohomology groups of $X_{\mathfrak{p}}$ are torsion-free. We note that this choice of N may depend on the choice of a regular proper model for X over $\text{Spec}(\mathcal{O}_{K,S})$. If \mathfrak{p} is such that $H^i(X_{\mathfrak{p}}, W\Omega_{X_{\mathfrak{p}}}^j)$ are all finite type, then by the degeneration of the slope spectral sequence at the E_1 -stage by Bloch-Nygaard (see Theorem 3.7 of [17]), and the fact that the crystalline cohomology groups are torsion free, it follows that the Hodge-Witt groups are free as well. If, on the other hand, some Hodge-Witt group of $X_{\mathfrak{p}}$ is not of finite type over W , then we are in the second case. \square

Question 5.1.2 (Joshi-Rajan). *Let X/K be a smooth projective variety over a number field. When does there exist an infinite set of primes of K such that the Hodge-Witt cohomology groups of the reduction $X_{\mathfrak{p}}$ at \mathfrak{p} are not Hodge-Witt?*

We would like to explicate the information encoded in such a set of primes (when it exists).

Proposition 5.1.3 (Joshi-Rajan). *Let A/K be an abelian surface over a number field K . Then there exists infinitely many primes \mathfrak{p} such that $H^2(X_{\mathfrak{p}}, W(\mathcal{O}_{X_{\mathfrak{p}}}))$ has infinite torsion if and only if there exists infinitely many primes \mathfrak{p} of supersingular reduction for X . In particular, let E be an elliptic curve over \mathbb{Q} and let $X = E \times_{\mathbb{Q}} E$. Then for an infinite set of primes of \mathbb{Q} , the Hodge-Witt groups $H^i(X_p, W\Omega_{X_p}^j)$ are not torsion free for $(i, j) \in \{(2, 0), (2, 1)\}$.*

Proof. The first part follows from the results of [17, Section 7.1(a)]. The second part follows from combining the first part with Elkies's theorem (see [13]), that given an elliptic curve E over \mathbb{Q} , there are infinitely many primes p of \mathbb{Q} such that E has supersingular reduction. \square

Remark 5.1.4. Question 5.1.2 which was raised by us in [21] can now formulated as a more precise conjecture (in the light of Corollary 4.4.8). Our formulation is the following.

Conjecture 5.1.5. *Let X/K be any smooth, projective variety of dimension n over a number field K . Assume that $H^n(X, \mathcal{O}_X) \neq 0$. Then there exists infinitely many primes \mathfrak{p} of K such that the domino associated to the differential $H^n(X_{\mathfrak{p}}, W(\mathcal{O}_{X_{\mathfrak{p}}})) \rightarrow H^n(X_{\mathfrak{p}}, W\Omega_{X_{\mathfrak{p}}}^1)$ is non-zero (here $X_{\mathfrak{p}}$ is the “reduction” of X at \mathfrak{p}). In particular X has non-Hodge-Witt reduction at \mathfrak{p} .*

Remark 5.1.6. Let us remind the reader that the domino (see [20]) associated to a de Rham-Witt differential such as $H^n(X_{\mathfrak{p}}, W(\mathcal{O}_{X_{\mathfrak{p}}})) \rightarrow H^n(X_{\mathfrak{p}}, W\Omega_{X_{\mathfrak{p}}}^1)$ is a measure of infinite torsion in the slope spectral sequence. In particular the conjecture says that for the reduction, $H^n(X_{\mathfrak{p}}, W(\mathcal{O}_{X_{\mathfrak{p}}}))$ has infinite torsion for infinitely many primes \mathfrak{p} . Let us understand what Conjecture 5.1.5 means in the context of a K3 surface. Let X/K be a smooth, projective K3 surface over a number field K . As $H^2(X, \mathcal{O}_X) \neq 0$, the hypothesis of Conjecture 5.1.5 is valid. Now X has Hodge-Witt reduction at a prime \mathfrak{p} of K if and only if $X_{\mathfrak{p}}$ is of finite height (i.e. the formal group attached to $X_{\mathfrak{p}}$ is of finite height). So $X_{\mathfrak{p}}$ has non-Hodge-Witt reduction if and only if the formal group attached to $X_{\mathfrak{p}}$ is of infinite height. In this case for the reduction one has $H^2(X_{\mathfrak{p}}, W(\mathcal{O}_{X_{\mathfrak{p}}})) = k[[V]]$ with $F = 0$ (as a module over the Cartier-Dieduonne algebra—see [17]).

Remark 5.1.7. Let us remark that Conjecture 5.1.5 implies Elkies’ Theorem on infinitude of primes of supersingular reduction *valid for any number field* (and not just \mathbb{Q}). To see this let E/K be an elliptic curve over any number field K . Let $A = E \times_K E$. Then A is an abelian surface and hence $H^2(A, \mathcal{O}_A) \neq 0$ and so Conjecture 5.1.5 predicts that there are infinitely many primes \mathfrak{p} of K such that A has non-Hodge-Witt reduction at \mathfrak{p} . But A has non-Hodge-Witt reduction if and only if the p -rank of A is zero. This means E has p -rank zero at all such \mathfrak{p} . Thus E has supersingular reduction at \mathfrak{p} . Hence E has supersingular reduction at infinitely many primes \mathfrak{p} of K .

Remark 5.1.8. Let us also remark that Conjecture 5.1.5 also implies that if E_1, E_2 are two elliptic curves over number fields, then there are infinitely many primes p (of the number field) where both E_1, E_2 have supersingular reduction. The set of such primes is, of course, expected to be very thin.

Remark 5.1.9. Let us note that even for pairs of elliptic curves over \mathbb{Q} with small conductors, the first prime of common supersingular reduction can be very large. Here is a particularly interesting example with “modular flavor.” Consider the modular curve $X_0(37)$, which has genus two and its Jacobian, $J_0(37)$, is isogenous to a product of two elliptic curves of conductor $N = 37$. The curves were described by Mazur and Swinnerton-Dyer in their classic paper on Weil curves and the two elliptic curve factors of $J_0(37)$ are not isogenous to each other. The two curves appear as 37a and 37b1 in John Cremona’s tables. So to find a prime of non-Hodge-Witt reduction for $J_0(37)$ is the same as finding a prime of common supersingular reduction for the elliptic curve factors of $J_0(37)$. The first such a prime is $p = 18489743$ (a number several order of magnitudes larger than the conductor). This makes searching for such primes for pairs of elliptic curves rather tedious, at least with the miniscule computational resources available to us, but the evidence for the conjecture, gathered using [43], is quite convincing despite this.

In the direction of the Conjecture 5.1.5 we have the following:

Theorem 5.1.10. *Let X be one of the following:*

- (1) *a Fermat hypersurface over \mathbb{Q} , or*
- (2) *an abelian variety with complex multiplication over some number field K .*

Then Conjecture 5.1.5 holds for X .

Proof. The assertion (1) is due to Joshi-Rajan (see [21] and uses results of [47]). The second assertion, after [44, Theorem 1.1], is reduced to a Chebotarev argument similar to the one

used in the proof of Theorem 4.4.5. To be precise one needs to show that there are infinitely many primes p such that at least two of the primes $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ lying over p in K_0 remains inert in K (which is clear by the method of proof of Theorem 4.4.5). \square

Remark 5.1.11. These are the only examples of this phenomena we know so far related to the above question.

6. SOME EXPLICIT EXAMPLES

In this subsection we describe some explicit examples of the diverse phenomena which the reader may find instructive or insightful.

6.1. Fermat varieties. Let us begin with Fermat varieties and explicitly compute densities of primes of various types of reduction. The examples we consider here are general type. Let $F_{m,n}$ be the closed subscheme of \mathbb{P}^{n+1} defined by the homogeneous equation

$$F_{n,m} : X_0^m + \dots + X_{n+1}^m = 0.$$

The notation is such that $F_{n,m}$ has dimension n . Let $\delta_{n,m}^{\text{ord}}$, $\delta_{m,n}^{\text{hw}}$, and $\delta_{n,m}^{\text{non-hw}}$ denote the asymptotic density of primes of ordinary, Hodge-Witt and non-Hodge-Witt reduction for $F_{n,m}/\mathbb{Q}$ (we ignore primes of bad reduction). Then the densities may be calculated by using the results of [47]. Except for a finite list of exceptional $F_{n,m}$, which are given in the table below, the densities are given by the following (we write $\delta_{n,m}^?$ for $\delta^?(F_{n,m}, \mathbb{Q})$, with $? = \text{ord}, \text{hw}, \text{non} - \text{hw}$):

$$(6.1.1) \quad \delta_{n,m}^{\text{ord}} = \frac{1}{\phi(m)}$$

$$(6.1.2) \quad \delta_{n,m}^{\text{hw}} = \frac{1}{\phi(m)}$$

$$(6.1.3) \quad \delta_{n,m}^{\text{non-hw}} = 1 - \frac{1}{\phi(m)}$$

In the table we list the density of primes of different types of reduction for $F_{n,m}$, we also list the condition for a prime so that $F_{n,m}$ has the reduction of the given sort. The table also shows that the density of ordinary and Hodge-Witt reductions can be different. In particular for the septic Fermat surface $F_{2,7} \subset \mathbb{P}^3$ the densities are $\delta_{2,7}^{\text{ord}} = \frac{1}{6} < \delta_{2,7}^{\text{hw}} = \frac{1}{2} = \delta_{2,7}^{\text{non-hw}}$. So the septic Fermat surface shows rather interesting behavior. We do not know why this surface shows this exceptional behavior.

(n, m)	ordinary $\delta_{n,m}^{\text{ord}}$	Hodge-Witt $\delta_{n,m}^{\text{hw}}$	non Hodge-Witt $\delta_{n,m}^{\text{non-hw}}$
$(1, m)$?	1	0
$(n, 1)$	1	1	0
$(n, 2)$	1	1	0
$(2, 3)$	$\frac{1}{2}$	1	0
$(3, 3)$	$\frac{1}{2}, (p \equiv 1 \pmod{3})$	1	0
$(3, 4)$	$\frac{1}{2}, (p \equiv 1 \pmod{4})$	1	0
$(5, 3)$	$\frac{1}{2}, (p \equiv 1 \pmod{3})$	1	0
$(2, 7)$	$\frac{1}{6}, (p \equiv 1 \pmod{7})$	$\frac{1}{2}, (p \equiv 1, 2, 4 \pmod{3})$	$\frac{1}{2}, (p \equiv 3, 5, 6 \pmod{7})$
(n, m)	$\frac{1}{\phi(m)}$	$\frac{1}{\phi(m)}$	$1 - \frac{1}{\phi(m)}$

Example 6.1.1. [Product of two elliptic curves] Here is another example of a similar sort. Let $E/\mathbb{Q} : y^2 = x^3 - x$ and $E'/\mathbb{Q} : y^2 = x^3 + 1$ be elliptic curves over \mathbb{Q} with complex multiplication by $\mathbb{Q}(i)$ and $\mathbb{Q}(\zeta_3)$ respectively. Then E has ordinary reduction if and only if $p \equiv 1 \pmod{4}$ and E' has ordinary reduction if and only if $p \equiv 1 \pmod{3}$. Let $A = E \times E'$. This is an abelian surface, and we want to calculate the densities δ_A^{ord} and δ_A^{hw} or primes of ordinary reduction and Hodge-Witt reduction for A . We consider primes p for which A has good reduction. Now suppose p is a prime of good reduction for A . Then we see by considering Newton and Hodge polygons of $A = E \times E'$ that A has ordinary reduction at p if and only if both E and E' have ordinary reductions at p (so $p \equiv 1 \pmod{3}$ and $p \equiv 1 \pmod{4}$). Thus we see that $\delta_A^{\text{ord}} = \frac{1}{4}$. While A has Hodge-Witt reduction if and only if one of E, E' is ordinary (curves always have Hodge-Witt reduction, so E, E' are always Hodge-Witt). Thus this happens if and only if p is 1 mod 4 or 1 mod 3. Thus we see that $\delta_A^{\text{hw}} = \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = \frac{3}{4}$. Thus we see that for Abelian surfaces it is possible to have $\delta_A^{\text{ord}} < \delta_A^{\text{hw}} < 1$.

Example 6.1.2. [CM by a field galois (over \mathbb{Q})] Let $C : y^2 = x^5 - 1$. Here the genus $g = 2$ and the Jacobian of C has CM by $\mathbb{Q}(\zeta_5)$ which is cyclic and galois. Then [14] its Jacobian is ordinary if and only if $p \equiv 1 \pmod{5}$ and for all other primes $p > 5$, the reduction is non-Hodge-Witt. In particular the set of prime of ordinary (=Hodge-Witt) reduction has density $\frac{1}{\phi(5)}$ (where $\phi(5) = |\text{Gal}(\mathbb{Q}(\zeta_5)/\mathbb{Q})| = 4$) and the set of primes of non-Hodge-Witt reduction is $1 - \frac{1}{\phi(5)} = \frac{4}{5}$.

Example 6.1.3. [CM by a non galois CM field] Consider the field $K = \mathbb{Q}[x]/(x^4 + 134x^2 + 89)$. This is a quartic non-galois CM field, its normal closure N has $G = \text{Gal}(N/\mathbb{Q}) = D_4$ the dihedral group of order 8. This field and abelian surfaces with CM by this field are discussed in [14]. Examining the conjugacy classes of G we see that the conjugacy class of transpositions has two elements, and the conjugacy class of the identity is of course one element. Now there exists an abelian variety A over some number field L , with CM by K . For A , by Theorem 4.4.5 the set of primes of Hodge-Witt reduction has density $\frac{3}{4} \geq \delta^{\text{hw}} \geq \frac{3}{8}$ while the density of primes of ordinary reduction is $\frac{1}{2} \geq \delta^{\text{ord}} \geq \frac{1}{8}$. We note that some primes other than the almost split primes contribute to the densities, and one can, using [14], describe them explicitly, but we do not work this out here—as this sort of calculation should be worked out in some generality and this will require some additional combinatorics which belongs to a separate paper by itself. For all other sufficiently large primes, the reduction is non-Hodge-Witt and there are infinitely many of these as well (from the above bounds).

Example 6.1.4 (Surfaces of general type). Suppose C, C' are smooth projective curves over \mathbb{Q} . Assume that C is hyperelliptic curve of genus $g \geq 2$ whose Jacobian has Complex multiplication and that genus of C' is at least two, the Jacobian of C (resp. C') is not isogenous to a factor of the Jacobian of C' (resp. C). Then C has good ordinary reduction at a positive density of primes. Now consider the surface $S = C \times_{\mathbb{Q}} C'$, then S is a surface with good Hodge-Witt reduction at positive density set of primes; and S has good ordinary reduction precisely when both C, C' have good ordinary reductions. Thus we have examples of surfaces of general type where Conjecture 4.1.1 and Conjecture 3.1.1 hold and provide different sets of primes.

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